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ASYMPTOTIC DENSITY OF STATES OF P-BRANES

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Abstract

The asymptotic form of the density of states for a p-brane is computed (using the semiclassical approximation for the expression of the mass formula), and a value is found which only resembles the exponential growth of black holes in the formal limit $p \to \infty$. Some physical consequences for the thermodinamics of these objects are also presented.

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1 Introduction

It is well known that the asymptotic density of states for strings grows exponentially, that is

$$\rho(E) = E^{-a} \exp(bE)\,, \tag{1}$$

where a and b are model dependent, positive constants. This generic fact has well-known consequences for the thermodynamics of strings (cf. ref. [1] for a review), of which a notorious example is the presence of a critical (Hagedorn) temperature above which strings cannot be in thermal equilibrium. The physical interpretation of this critical temperature is not yet clear, although it can possibly signal the presence of a phase transition to a completely different phase (cf., for example, [2]).

Higher dimensional expended objects (p-branes) have sometimes been proposed as interesting natural generalizations of strings (cf., for example, [3]). One naturally grows curious as to what the density of states of these objects looks like. A natural intuition is that we have got many more states than in strings, so that we expect the density to grow even faster, perhaps a natural first guess could be

$$\rho(E) = E^{-a} \exp(bE^p)\,. \tag{2}$$

A difficulty that appears as soon as detailed computations are attempted is that there is not yet a consistent quantum theory of these objects. A natural first approximation is the semiclasical quantization, which has been studied in the reference [4] for membranes propagating in flat toruses (cf. also the paper [5]).

According to this approximation, (and neglecting winding states, presumably inessential for the purposes of this letter), the mass formula for p-branes compactified in $(S^1)^p \times R^{d-p}$ is expressed as a generalization of the corresponding formula for strings, namely:

$$M^2 = \sum_{N_{n_1 \ldots n_p}} \omega_{n_1 \ldots n_p} . N_{n_1 \ldots n_p}\,, \tag{3}$$

where the frequencies are given by

$$\omega^2_{n_1 \ldots n_p} = \pi \sum_{i=1}^{p} (n_i/a_i)^2\,, \tag{4}$$

where $a_1, ..., a_p$ are the radius of the different circles of the cartesian torus.

It should be kept in mind that the consistency of the semiclassical approximation is by no means obvious. It has been suggested , in particular, that membranes can be obtained as a sort of $N \to \infty$ limit (albeit a very complicated one) of a family of $SU(N)$ matrix models in quantum mechanics (cf. [3]). In this way it has been claimed that the spectrum of the regulated supermembranes is a continuous one; the existence in it of a massless sector is, however, dubious. Super p-branes are, finally, classically unstable with respect to collapse into lower dimensional configurations.

In this paper, we shall stick, however, to the semiclassical approximation, and we shall take it as defining a model by itself; a natural generalization, on the other hand, of the mass spectrum for strings. We shall eventually find interesting analogies with other physical systems.

2 The asymptotic density of states of a p-brane

The quantity of interest is

$$\Omega(M^2) = \sum_{N_{n_1 \ldots n_p}} \theta(M^2 - \sum N_{n_1 \ldots n_p} \omega_{n_1 \ldots n_p}) \,. \tag{5}$$

In order to get an estimate of the above quantity, analytic methods will be needed; the frequencies not being natural numbers except for strings ($p = 1$). We first estimate the contribution coming from a fixed number A of frequencies, which we assume ordered,$\omega_1 \leq \omega_2 \leq \ldots \leq \omega_A$

$$\Omega_A(M^2) = \sum_N \theta(M^2 - N_1\omega_1 - \ldots - N_A\omega_A) \,, \tag{6}$$

which we will do by approaching the sum by an integral,

$$\int_0^\infty dx_1 \ldots dx_A \theta(M^2 - x_1\omega_1 - \ldots - x_A\omega_A) \,. \tag{7}$$

This gives inmediatly

$$\Omega_A(M^2) = M^{2A}/(A!\omega_1 \ldots \omega_A) \,. \tag{8}$$

We now have to face the task of estimating the sum over A; symbollically:

$$\Omega(M^2) = \sum_A \Omega_A(M^2) \,. \tag{9}$$

In order to do that in the simplest possible way, let us consider the probability density $\rho(\omega)d\omega$ of the allowed frequencies; that is, a function such that the total number of frequencies smaller than or equal to a given one,ω, is given by:

$$N(\omega) = \int_0^\omega dx \rho(x) \,. \tag{10}$$

We have

$$\rho(\omega) = \int dx_1 ... dx_p \delta(\omega - (x_1^2 + ... + x_p^2)^{1/2}) \sim \alpha \omega^{p-1} \,, \tag{11}$$

which yields a behavior for the distribution of frequencies of:

$$N(\omega) \sim \alpha / p \omega^p \,. \tag{12}$$

This, in turn, means that the frequency which has got the number i in the ordering just introduced is typically of the order

$$i = \alpha / p \omega_i^p \,, \tag{13}$$

so that a good estimate of the product of the first A frequencies is given by

$$\omega_1 ... \omega_p \sim (1.2....A)^{1/p} = (A!)^{1/p} \,. \tag{14}$$

We have now reduced the task to the estimate of

$$\Omega(M^2) \sim \sum_A M^{2A} / (A!)(A!)^{-1/p} \,, \tag{15}$$

and by approaching again this sum by an integral, and performing a saddle point approximation, we get

$$\Omega(M^2) \sim \exp(\lambda M^{2p/(p+1)}) \,, \tag{16}$$

where λ is a numerical constant which in this approximation takes on the value: $\lambda = (p+1)/pe$. This means that the rough intuitive estimate we nade in the introduction was grossly wrong, and that p-branes have a density of states which never grows as fast as $\exp(M^2)$.

3 Physical implications and conclusions

Super p-branes need, in order to have a matching number of bosonic and fermionic degrees of freedom, the presence of the famous κ-symmetry, which in turn selects only a small number of p-branes moving in dimension d. To be specific, the allowed values of (p, d) are $(0, d), (1, 3), (1, 4), (1, 6), (1, 10), (2, 4), (2, 5), (2, 7), (2, 11), ($

This means that it makes no physical sense to take the $\lim_{p \to \infty}$ in the formulas as above. It is nevertheless curious that only in this limit we get the behaviour

$$\rho(M^2) \sim \exp(M^2)\,, \tag{17}$$

characteristic of four-dimensional black-holes.

In all physical instances of p-branes (that is, for $1 < p < 5$), we get a behaviour which fits in between strings and black holes: the density of states grows quicker than for ordinary strings (which means, in particular, that there is never canonical equilibrium for these systems, at least in the ordinary sense, or if one wants to put it in other terms, that the Hagedorn temperature $T_c = 0$); but it grows slower than for ordinary black holes.

Assuming that all states on the constant energy surface are equally probable, one can generalize the classical argument of Hawking [6] for the microcanonical distribution of a p-brane with a photon gas. If we represent by E the total energy of the system, and by E_r the part of the energy carried by the photons, we easily get the result that equilibrium is possible only if the total volume of the system is smaller than a critical value, given by:

$$V_c = K(E - E_r)^{-d(p+2)/(p+1)} E_r^{d+1}\,, \tag{18}$$

(where K is a numerical constant).

In conclusion , clearly more work is needed before the main quantum characteristics of (super)p-branes are unveiled. We have, however, discovered an intriguing charateristic of their spectrum (as computed in the semiclassical approximation), namely, that it grows much smoother than previously thought.

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Note added

S. Odintsov (private communication) has checked that the inclusion of winding modes does not change qualitatively the conclusions of the present paper.

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